OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS — 2605

Pure Mathematics 5

Thursday **10 JANUARY 2002** Morning 1 hour 20 minutes

Additional materials:
- Answer booklet
- Graph paper
- MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 The equation $x^4 - 6x^3 - 73x^2 + kx + m = 0$ has two positive roots α, β and two negative roots γ, δ. It is given that $\alpha\beta = \gamma\delta = 4$.

(i) Find the values of the constants k and m. [5]

(ii) Show that $(\alpha + \beta)(\gamma + \delta) = -81$. [4]

(iii) Find the quadratic equation which has roots $\alpha + \beta$ and $\gamma + \delta$. [2]

(iv) Find $\alpha + \beta$ and $\gamma + \delta$. [3]

(v) Show that $\alpha^2 - 3(1+\sqrt{10})\alpha + 4 = 0$, and find similar quadratic equations satisfied by β, γ and δ. [6]

2 **(a)** By considering $(\cos\theta + \mathrm{j}\sin\theta)^5$, show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$ [5]

(b) **(i)** Find the modulus and argument of $-8 + 8\sqrt{3}\,\mathrm{j}$, and hence find the fourth roots of $-8 + 8\sqrt{3}\,\mathrm{j}$ in the form $r\mathrm{e}^{\mathrm{j}\theta}$ with $r > 0$ and $-\pi < \theta \leqslant \pi$. [6]

The points representing these fourth roots on an Argand diagram are the vertices of a square. These vertices are labelled A, B, C, D starting with A in the first quadrant and going anticlockwise.

(ii) Draw an Argand diagram showing the square ABCD.

Find the length of a side of this square. [4]

The midpoints of the sides AB, BC, CD, DA represent complex numbers z_1, z_2, z_3, z_4, which are the fourth roots of a complex number w.

(iii) By first finding the modulus and argument of z_1, or otherwise, find w, giving your answer in the form $a + b\mathrm{j}$. [5]

3 **(i)** Prove that $\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$. [5]

(ii) Find $\displaystyle\int_0^2 \frac{1}{\sqrt{3x^2+4}}\,\mathrm{d}x$, giving your answer in logarithmic form. [5]

(iii) Find the exact value of $\displaystyle\int_0^2 \frac{1}{3x^2+4}\,\mathrm{d}x$. [4]

(iv) Use the substitution $x\sqrt{3} = 2\tan\theta$ to show that $\displaystyle\int_0^2 \frac{1}{(3x^2+4)^{\frac{3}{2}}}\,\mathrm{d}x = \frac{1}{8}$. [6]

4 **(a)** Sketch the conic with polar equation $\frac{a}{r} = 1 + \cos\theta$.

Draw the directrix on your diagram, and give the equation of the directrix in the form $r\cos\theta = k$. [5]

(b) The ellipse E has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Two points P and Q on E have coordinates $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$.

(i) Show that the equations of the tangents at P and Q are

$$(b\cos\theta)x + (a\sin\theta)y = ab,$$
$$(-b\sin\theta)x + (a\cos\theta)y = ab.$$ [6]

These two tangents meet at the point T.

(ii) Find the coordinates of T. Show that T lies on the ellipse F with equation

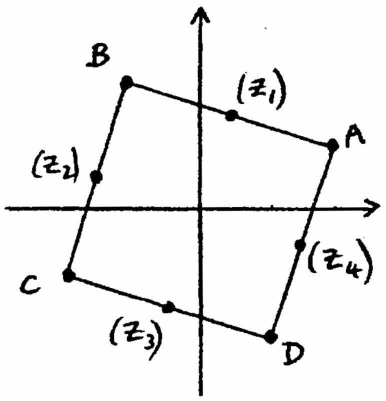
$$\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1.$$ [7]

(iii) Describe fully the geometrical transformation which transforms the ellipse E into the ellipse F. [2]

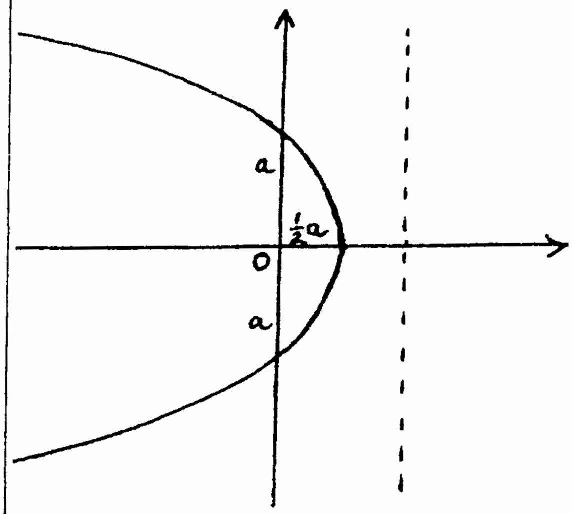
Mark Scheme

MARKING SCHEME

1 (i)	$\alpha+\beta+\gamma+\delta=6$ $k=-(\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma)$ $=-4(\beta+\alpha+\delta+\gamma)$ $=-24$ $m=\alpha\beta\gamma\delta=16$	B1 M1 M1 A1 B1 **5**	 *Condone sign error* Using $\alpha\beta=\gamma\delta=4$
(ii)	$\sum\alpha\beta=-73$ $(\alpha+\beta)(\gamma+\delta)=\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta$ $=\sum\alpha\beta-\alpha\beta-\gamma\delta$ $=-73-4-4$ $=-81$	B1 M1A1 A1 (ag) **4**	
(iii)	$\alpha+\beta,\ \gamma+\delta$ have sum 6 and product -81 so they are the roots of $y^2-6y-81=0$	M1 A1 **2**	
(iv)	$y=\dfrac{6\pm\sqrt{360}}{2}=3\pm3\sqrt{10}$ Since α,β are positive, $\alpha+\beta=3+3\sqrt{10}$ $\gamma+\delta=3-3\sqrt{10}$	M1A1 ft A1 ft **3**	
(v)	α,β have sum $3+3\sqrt{10}$ and product 4 so they are the roots of $z^2-3(1+\sqrt{10})z+4=0$ Hence $\alpha^2-3(1+\sqrt{10})\alpha+4=0$ $\beta^2-3(1+\sqrt{10})\beta+4=0$ γ,δ have sum $3-3\sqrt{10}$ and product 4 Hence $\gamma^2-3(1-\sqrt{10})\gamma+4=0$ $\delta^2-3(1-\sqrt{10})\delta+4=0$	M1 A1 (ag) A1 M1 A1 ft A1 ft **6**	

2 (a)	$\cos 5\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5$ $= \cos^5\theta + 5\cos^4\theta(j\sin\theta) + 10\cos^3\theta(j\sin\theta)^2 + \ldots$ $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \quad [c = \cos\theta]$ $= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	M1 M1 A1 M1 A1 (ag) 5	Expanding $(\cos\theta + j\sin\theta)^5$ $\cos 5\theta = \mathrm{Re}(\cos\theta + j\sin\theta)^5$
(b)(i)	$\lvert -8 + 8\sqrt{3}j \rvert = 16$, $\arg(-8 + 8\sqrt{3}j) = \frac{2}{3}\pi$ Fourth roots have $r = 2$ $\theta = \frac{1}{6}\pi,\ \frac{2}{3}\pi,\ -\frac{1}{3}\pi,\ -\frac{5}{6}\pi$	B1B1 B1 ft B3 6	Or 120° Give B1 (ft) for one correct Give B2 (ft) for two correct *Max B2 if given in degrees or if some values out of range*
(ii)	 Length of a side is $\sqrt{2^2 + 2^2} = \sqrt{8}$	B2 M1A1 ft 4	Square with correct orientation, diagonals passing through O. Give B1 (ft) if approx correct.
(iii)	z_1 has modulus $\frac{1}{2}\sqrt{8} = \sqrt{2}$ and argument $\frac{1}{2}(\frac{1}{6}\pi + \frac{2}{3}\pi) = \frac{5}{12}\pi$ $w = z_1^4 = 4(\cos\frac{5}{3}\pi + j\sin\frac{5}{3}\pi)$ $= 2 - 2\sqrt{3}j$	B1 ft B1 ft M1A1 ft A1 5	
	OR $z_1 = \frac{1}{2}(\sqrt{3} - 1) + \frac{1}{2}(\sqrt{3} + 1)j$ — B1 ft $z_1^2 = -\sqrt{3} + j$ — M1 $w = z_1^4 = 2 - 2\sqrt{3}j$ — M1A1A1 cao		

3 (i)	$x=\frac{1}{2}(e^{y}-e^{-y})$ (where $y=\operatorname{arsinh} x$)	M1	
	$e^{2y}-2xe^{y}-1=0$	M1	
	$e^{y}=\dfrac{2x\pm\sqrt{4x^2+4}}{2}\quad\left(=x\pm\sqrt{x^2+1}\right)$	M1	
	$e^{y}=x+\sqrt{x^2+1}$ (since $e^{y}>0$)	A1	
	$y=\ln(x+\sqrt{x^2+1})$	A1 (ag) **5**	*Not dependent on previous A1*
	OR If $x=\sinh y$ then $\sqrt{x^2+1}=\cosh y$ M1 $x+\sqrt{x^2+1}=\sinh y+\cosh y$ M1 $=\frac{1}{2}(e^{y}-e^{-y})+\frac{1}{2}(e^{y}+e^{-y})=e^{y}$ M1A1 $\operatorname{arsinh} x=y=\ln(x+\sqrt{x^2+1})$ A1		
(ii)	$\displaystyle\int_0^2\frac{1}{\sqrt{3x^2+4}}\,dx=\left[\frac{1}{\sqrt{3}}\operatorname{arsinh}\frac{\sqrt{3}x}{2}\right]_0^2$	M1 A1	For arsinh, *or* any sinh substitution For $\operatorname{arsinh}\dfrac{\sqrt{3}x}{2}$ *or* $\sqrt{3}x=2\sinh\theta$
	$=\dfrac{1}{\sqrt{3}}\operatorname{arsinh}\sqrt{3}$	A1	For factor $\dfrac{1}{\sqrt{3}}$ *or* $\displaystyle\int\frac{1}{\sqrt{3}}\,d\theta$
	$=\dfrac{1}{\sqrt{3}}\ln(\sqrt{3}+2)$	M1 A1 **5**	Using $\operatorname{arsinh} x=\ln(x+\sqrt{x^2+1})$
	OR M2 $\displaystyle\int_0^2\frac{1}{\sqrt{3x^2+4}}\,dx=\left[\frac{1}{\sqrt{3}}\ln\left(\sqrt{3}x+\sqrt{3x^2+4}\right)\right]_0^2$ A1A1 $=\dfrac{1}{\sqrt{3}}\left(\ln(2\sqrt{3}+4)-\ln 2\right)$ $=\dfrac{1}{\sqrt{3}}\ln(\sqrt{3}+2)$ A1		Integral of form $\ln\left(ax+\sqrt{a^2x^2+b^2}\right)$ For $\dfrac{1}{\sqrt{3}}$ and $\ln\left(\sqrt{3}x+\sqrt{3x^2+4}\right)$ (*or* $\ln\left(x+\sqrt{x^2+\frac{4}{3}}\right)$)
(iii)	$\displaystyle\int_0^2\frac{1}{3x^2+4}\,dx=\left[\frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}x}{2}\right]_0^2$	M1 A1	For arctan, *or* any tan substitution For $\arctan\dfrac{\sqrt{3}x}{2}$ *or* $\sqrt{3}x=2\tan\theta$
	$=\dfrac{1}{2\sqrt{3}}\arctan\sqrt{3}$	A1	For factor $\dfrac{1}{2\sqrt{3}}$ *or* $\displaystyle\int\frac{1}{2\sqrt{3}}\,d\theta$
	$=\dfrac{\pi}{6\sqrt{3}}$	A1 **4**	
(iv)	$\displaystyle\int_0^2\frac{1}{(3x^2+4)^{\frac{3}{2}}}\,dx=\int_0^{\frac{1}{3}\pi}\frac{1}{8\sec^3\theta}\,\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$	M1A1	For $\dfrac{1}{8\sec^3\theta}$
	$=\dfrac{1}{4\sqrt{3}}\displaystyle\int_0^{\frac{1}{3}\pi}\cos\theta\,d\theta$	B1	For $\dfrac{2}{\sqrt{3}}\sec^2\theta$
	$=\dfrac{1}{4\sqrt{3}}\left[\sin\theta\right]_0^{\frac{1}{3}\pi}$	M1 B1	Integrating $\cos\theta$ Substituting $\theta=\frac{1}{3}\pi$
	$=\dfrac{1}{4\sqrt{3}}\,\dfrac{\sqrt{3}}{2}=\dfrac{1}{8}$	A1 (ag) **6**	

4 (a)		B1	Parabola with correct orientation
		B1	Positive intercept on x-axis, symmetrical intercepts on y-axis, with at least one intercept indicated
		B1 ft	Directrix in (approx) correct position
		M1	Directrix and O at same distance from vertex (implied by $x = a$)
	Equation of directrix is $r\cos\theta = a$	A1 **5**	
(b)(i)	$\dfrac{dy}{dx} = \dfrac{b\cos\theta}{-a\sin\theta}$	M1 A1	Parametric or implicit differentiation *These marks M1A1M1 can be awarded for Q if not earned for P*
	Tangent at P is $y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$	M1	
	$(b\cos\theta)x + (a\sin\theta)y = ab\cos^2\theta + ab\sin^2\theta$ $(b\cos\theta)x + (a\sin\theta)y = ab$	A1 (ag)	
	Tangent at Q is $y - b\cos\theta = \dfrac{-b\sin\theta}{-a\cos\theta}(x + a\sin\theta)$	M1	Complete method for second tangent e.g. replacing θ by $(\theta + \frac{1}{2}\pi)$
	$(-b\sin\theta)x + (a\cos\theta)y = ab\cos^2\theta + ab\sin^2\theta$ $(-b\sin\theta)x + (a\cos\theta)y = ab$	A1 (ag) **6**	
(ii)	Solving simultaneously, $(b\cos^2\theta + b\sin^2\theta)x = ab(\cos\theta - \sin\theta)$	M1	Eliminating one variable
	$x = a(\cos\theta - \sin\theta)$	A1	
	$(a\sin^2\theta + a\cos^2\theta)y = ab(\sin\theta + \cos\theta)$	M1	Finding second variable
	$y = b(\sin\theta + \cos\theta)$	A1	
	$\dfrac{x^2}{2a^2} + \dfrac{y^2}{2b^2} = \frac{1}{2}\left\{(\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2\right\}$	M1	Substituting coordinates of T
	$= \frac{1}{2}\left(2\cos^2\theta + 2\sin^2\theta\right) = 1$	M1A1 (ag) **7**	
	OR from equations of tangents, $((b\cos\theta)x + (a\sin\theta)y)^2 + ((-b\sin\theta)x + (a\cos\theta)y)^2 = 2a^2b^2$ M1 $b^2x^2 + a^2y^2 = 2a^2b^2$ M1 $\dfrac{x^2}{2a^2} + \dfrac{y^2}{2b^2} = 1$ A1		
(iii)	F has semi axes $a\sqrt{2}$ and $b\sqrt{2}$ Transformation is an enlargement, centre O,	B1	
	scale factor $\sqrt{2}$	B1 **2**	

Examiner’s Report

Pure Mathematics 5 (2605)

General Comments

The number of candidates taking this paper was significantly lower than in previous January sessions, but the quality of the work was generally good; about a third of the candidates scored 50 marks or more (out of 60). Most candidates appeared to have sufficient time to complete the paper. Q.4 (on conics) was much less popular than the other questions.

Comments on Individual Questions

Question 1 (Roots of a Quartic Equation)

This was the best answered question, with half the attempts scoring 17 marks or more (out of 20), and about a quarter scoring full marks. Parts (i) to (iv) were well understood, although many sign errors were made. Part (v) was sometimes omitted, but those who attempted it usually understood what was required.

[(i) $k=-24,\ m=16$; (iii) $y^2-6y-81=0$; (iv) $\alpha+\beta=3+3\sqrt{10}$, $\gamma+\delta=3-3\sqrt{10}$;
(v) $\beta^2-3(1+\sqrt{10})\beta+4=0$, $\gamma^2-3(1-\sqrt{10})\gamma+4=0$, $\delta^2-3(1-\sqrt{10})\delta+4=0$]

Question 2 (Complex numbers)

This question was fairly well answered, and the average mark was about 13. In part (a) the identity was usually proved correctly. Parts (b)(i) and (ii) were quite well understood, but minor errors were often made, particularly with the arguments. In part (b)(iii) a lot of candidates made hard work of finding $|z_1|$, not realising that it was half the side of the square; and the correct value of w was rarely seen.

[(b)(i) $16,\ \frac{2}{3}\pi$; $r=2,\ \theta=\frac{1}{6}\pi,\ \frac{2}{3}\pi,\ -\frac{1}{3}\pi,\ -\frac{5}{6}\pi$; (ii) $\sqrt{8}$; (iii) $w=2-2\sqrt{3}\,\mathrm{j}$]

Question 3 (Calculus)

This was the most popular question, and it was well answered, with half the attempts scoring 16 marks or more. Most candidates seemed to be familiar with the proof in part (i), although the reason for rejecting $e^y = x - \sqrt{x^2+1}$ was not always convincingly explained. In parts (ii) and (iii) the forms of the integrals were almost always correct, but the factors $\frac{1}{\sqrt{3}}$ and $\frac{1}{2\sqrt{3}}$ were very often missing or incorrect. Part (iv) was quite often correctly done, although surprisingly many candidates were unable to proceed beyond the first step $\int \frac{1}{(4\tan^2\theta+4)^{\frac{3}{2}}}\frac{2}{\sqrt{3}}\sec^2\theta\,\mathrm{d}\theta$.

[(ii) $\frac{1}{\sqrt{3}}\ln(\sqrt{3}+2)$; (iii) $\frac{\pi}{6\sqrt{3}}$]

Question 4 (Conics)

This was attempted by only one third of the candidates, and it was by far the worst answered question. No candidate scored full marks, and half the attempts scored 6 marks or less. In part (a) many were unable to sketch the conic which was given in standard polar form. Finding the equations of the tangents in part (b)(i) was the only part of this question which was at all well understood, but even with the answers given there were many careless errors (especially of signs). The algebra required in part (b)(ii) defeated most candidates, and there were very few correct answers to the final part (b)(iii).

[(a) $r\cos\theta = a$; (b)(ii) $\left(a(\cos\theta-\sin\theta),\, b(\sin\theta+\cos\theta)\right)$;
(iii) Enlargement, centre O, scale factor $\sqrt{2}$]